Operational Characterization of Simultaneous Measurements in Quantum Mechanics

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(February 1, 2008)

Abstract

Quantum mechanics predicts the joint probability distribution of the outcomes of simultaneous measurements of commuting observables, but, in the state of the art, has lacked the operational definition of simultaneous measurements. The question is answered as to when the consecutive applications of measuring apparatuses give a simultaneous measurement of their observables. For this purpose, all the possible state reductions caused by measurements of an observable is also characterized by their operations.

PACS number: 03.65.Bz, 03.65.-w, 03.67.-a

In quantum mechanics, observables are represented by linear operators, for which the product operation is not commutative. If two observables are represented by commuting operators, they are simultaneously measurable and quantum mechanics predicts the joint probability distribution of the outcomes of their simultaneous measurement. But, it has not been answered fully what measurement can be considered as a simultaneous measurement of those observables.

Let $\mathbf{S}$ be a quantum system with the Hilbert space $\mathcal{H}$ of state vectors. Let A be an observable of $\mathbf{S}$. For any Borel set Δ in the real line R, the spectral projection of A corresponding to Δ is denoted by $E^A(\Delta)$; if A has the Dirac type spectral representation

$$A = \sum_{\nu} \sum_{\mu} \mu |\mu, \nu\rangle\langle\mu, \nu| + \sum_{\nu} \int \lambda |\lambda, \nu\rangle\langle\lambda, \nu|\, d\lambda, \tag{1}$$

where μ varies over the discrete eigenvalues, λ varies over the continuous eigenvalues, and ν is the degeneracy parameter, then we have

$$E^A(\Delta) = \sum_{\nu} \sum_{\mu\in\Delta} |\mu, \nu\rangle\langle\mu, \nu| + \sum_{\nu} \int_{\Delta} |\lambda, \nu\rangle\langle\lambda, \nu|\, d\lambda. \tag{2}$$

For any real number t, we shall denote by $\rho(t)$ the state (density operator) of $\mathbf{S}$ at the time t. We shall denote by "$A(t) \in \Delta$" the probabilistic event that *the outcome of the measurement (under consideration) of an observable A at the time t is in a Borel set* Δ. According to the Born statistical formula, we have

$$\Pr\{A(t) \in \Delta\} = \mathrm{Tr}[E^A(\Delta)\rho(t)]. \tag{3}$$

Any commuting observables A and B are simultaneously measurable and the joint probability distribution of the outcomes of their simultaneous measurement is postulated by

$$\Pr\{A(t) \in \Delta, B(t) \in \Delta'\} = \mathrm{Tr}[E^A(\Delta)E^B(\Delta')\rho(t)]. \tag{4}$$

A well-known proof of this quantum rule runs as follows [1]. Since A and B commute, there exist an observable C and Borel functions f and g such that $A = f(C)$ and $B = g(C)$. For the outcome c of the C-measurement, define the outcome of the A-measurement to be $f(c)$ and the outcome of the B-measurement to be $g(c)$. Then, it follows from (3) that their outcomes satisfy (4) so that the measurement of C at the time t gives a simultaneous measurement of A and B.

The above proof gives one special instance of simultaneous measurement which uses only one measuring apparatus, but it is rather open when a pair of measuring apparatuses for A and B make a simultaneous measurement of A and B. The purpose of this paper is to answer this question.

The conventional approach to the problem assumes the *projection postulate* formulated by Lüders [2] as follows: *If an observable A is measured in a state ρ, then at the time just after measurement the object is left in the state*

$$\frac{E^A(\{a\})\rho E^A(\{a\})}{\mathrm{Tr}[E^A(\{a\})\rho]},$$

provided that the object leads to the outcome a.

Suppose that at time t an observable A is measured by an apparatus satisfying the projection postulate and that at the time, $t+\Delta t$, just after the A-measurement an arbitrary observable B is measured. The joint probability distribution of the outcomes of the A-measurement and the B-measurement is given by

$$\begin{aligned}&\Pr\{A(t)\in\Delta, B(t+\Delta t)\in\Delta'\}\\&\quad=\sum_{a\in\Delta}\mathrm{Tr}[E^B(\Delta)E^A(\{a\})\rho(t)E^A(\{a\})].\end{aligned}\tag{5}$$

Thus, if A and B commute, we have

$$\Pr\{A(t)\in\Delta, B(t+\Delta t)\in\Delta'\}=\mathrm{Tr}[E^A(\Delta)E^B(\Delta')\rho(t)].\tag{6}$$

We can therefore reinterpret the B-measurement at the time $t+\Delta t$ as the B-measurement at the time t and consider the above consecutive measurements of A and B as the simultaneous measurement of A and B at the time t.

If we would restrict measurements to those satisfying the projection postulate, any consecutive measurements of commuting observables could be considered as simultaneous measurement, but this approach has the following limitations:

(i) Some of the most familiar measuring apparatuses such as photon counters do not satisfy the projection postulate [3].

(ii) When the observable has continuous spectrum, no measurements satisfy the projection postulate [4].

The above limitations appear indeed quite serious. From (i), we cannot apply (4) to correlation measurements using photon counters as in most of optical experiments and EPR-correlation measurements [5–8]. From (ii), we cannot use (4) for any measurements of continuous observables using more than one apparatuses.

Now we shall abandon the projection postulate and consider the following problem: *under what condition can consecutive measurements of two or more observables be considered as a simultaneous measurement of those observables?*

Suppose that an observable A of $\mathbf{S}$ is measured at time t by a measuring apparatus $\mathbf{A}$ with the state space $\mathcal{H}_{\mathbf{A}}$. Let $t+\Delta t$ be the time just after measurement. The measurement is carried out by the interaction between $\mathbf{S}$ and $\mathbf{A}$ from t to $t+\Delta t$, and after the time $t+\Delta t$ the object $\mathbf{S}$ is free from the apparatus $\mathbf{A}$. Suppose that the apparatus is in the state σ at the time t and let U be the unitary operator representing the time evolution of the composite system $\mathbf{S}+\mathbf{A}$ from t to $t+\Delta t$. Then, the object $\mathbf{S}$ is in the state

$$\rho(t+\Delta t)=\mathrm{Tr}_{\mathbf{A}}[U(\rho(t)\otimes\sigma)U^\dagger]\tag{7}$$

at the time $t+\Delta t$, where $\mathrm{Tr}_{\mathbf{A}}$ is the partial trace over $\mathcal{H}_{\mathbf{A}}$. The state change $\rho(t)\mapsto\rho(t+\Delta t)$ is determined independent of the outcome of measurement and called the *nonselective state change*.

Let B be an arbitrary observable of $\mathbf{S}$. We say that the measurement of A using the apparatus $\mathbf{A}$ *does not disturb* the observable B iff the nonselective state change does not perturb the probability distribution of B, that is, we have

$$\mathrm{Tr}[E^B(\Delta)e^{-iH\Delta t/\hbar}\rho(t)e^{iH\Delta t/\hbar}]=\mathrm{Tr}[E^B(\Delta)\rho(t+\Delta t)]\tag{8}$$

for any Borel set Δ, where H is the Hamiltonian of the system $\mathbf{S}$. The measurement is said to be *instantaneous* iff the duration Δt of measuring interaction is negligible in the time scale of the time evolution of the object. Thus, the instantaneous measurement of A using $\mathbf{A}$ does not disturb B if and only if

$$\mathrm{Tr}[E^B(\Delta)\rho(t)] = \mathrm{Tr}[E^B(\Delta)\rho(t+\Delta t)] \tag{9}$$

for any Borel set Δ.

By (7) and by the property of the partial trace, we have

$$\begin{aligned}
&\mathrm{Tr}[E^B(\Delta)\rho(t+\Delta t)] \\
&\quad = \mathrm{Tr}\left[E^B(\Delta)\mathrm{Tr}_{\mathbf{A}}[U(\rho(t)\otimes\sigma)U^\dagger]\right] \\
&\quad = \mathrm{Tr}\left[\mathrm{Tr}_{\mathbf{A}}[U^\dagger(E^B(\Delta)\otimes I)U(I\otimes\sigma)]\rho(t)\right].
\end{aligned} \tag{10}$$

Since $\rho(t)$ is arbitrary, (9) is equivalent to

$$E^B(\Delta) = \mathrm{Tr}_{\mathbf{A}}[U^\dagger(E^B(\Delta)\otimes I)U(I\otimes\sigma)] \tag{11}$$

for any Borel set Δ.

Now we can state precisely the answer to the above problem to be obtained in the present paper.

Theorem 1. *The instantaneous measurement of an observable A at time t using an apparatus $\mathbf{A}$ does not disturb an observable B if and only if we have*

$$\begin{aligned}
&\Pr\{A(t)\in\Delta, B(t+\Delta t)\in\Delta'\} \\
&\qquad = \mathrm{Tr}[E^A(\Delta)E^B(\Delta')\rho(t)]
\end{aligned} \tag{12}$$

for any density operator $\rho(t)$ and any Borel sets Δ and Δ'. In this case, A and B necessarily commute.

In what follows, we shall characterize all the possible state reductions caused by measurements of an observables in order to provide the proof of the above theorem,

For any Borel set Δ, let $\rho(t+\Delta t|A(t)\in\Delta)$ be the state at $t+\Delta t$ of the object $\mathbf{S}$ conditional upon $A(t)\in\Delta$. Thus, if the object $\mathbf{S}$ is sampled randomly from the subensemble of the similar systems that yield the outcome of the A-measurement in the Borel set Δ, then $\mathbf{S}$ is in the state $\rho(t+\Delta t|A(t)\in\Delta)$ at the time $t+\Delta t$. When $\Delta = \mathrm{R}$, the condition $A(t)\in\Delta$ makes no selection, and hence we have

$$\rho(t+\Delta t|A(t)\in\mathrm{R}) = \rho(t+\Delta t). \tag{13}$$

For $\Delta\neq\mathrm{R}$, the state change $\rho(t)\mapsto\rho(t+\Delta t|A(t)\in\Delta)$ is called the *selective state change*. When $\Pr\{A(t)\in\Delta\}=0$, the state $\rho(t+\Delta t|A(t)\in\Delta)$ is indefinite, and let $\rho(t+\Delta t|A(t)\in\Delta)$ be an arbitrarily chosen density operator for mathematical convenience.

Davies and Lewis [9] postulated that for any Borel set Δ there exists a positive linear transformation T_Δ on the space $\tau c(\mathcal{H})$ of trace class operators on $\mathcal{H}$ satisfying the following conditions:

(i) For any Borel set Δ and disjoint Borel sets Δ_n such that $\Delta = \bigcup_n \Delta_n$ and for any $\rho\in\tau c(\mathcal{H})$,

$$T_\Delta(\rho) = \sum_n T_{\Delta_n}(\rho). \tag{14a}$$

(ii) For any Borel set Δ and any $\rho \in \tau c(\mathcal{H})$,

$$\mathrm{Tr}[T_\Delta(\rho)] = \mathrm{Tr}[E^A(\Delta)\rho]. \tag{14b}$$

(iii) For any Borel set Δ with $\Pr\{A(t) \in \Delta\} > 0$,

$$\rho(t + \Delta t | A(t) \in \Delta) = \frac{T_\Delta[\rho(t)]}{\mathrm{Tr}[T_\Delta[\rho(t)]]}. \tag{14c}$$

The validity of the Davies-Lewis postulate was previously demonstrated in [4] *based on* the joint probability formula, where it is also shown that any such transformations T_Δ which are realizable by a measuring process are completely positive and *vice versa*. In what follows, we shall prove the Davies-Lewis postulate *without* assuming the joint probability formula so as to avoid the circular argument.

Suppose that at the time $t + \Delta t$ the observer were to measure an arbitrary observable B of the same object $\mathbf{S}$. Then, the joint probability distribution of the outcomes of the A-measurement and the B-measurement satisfies

$$\begin{aligned}&\Pr\{A(t) \in \Delta, B(t + \Delta t) \in \Delta'\}\\ &\quad = \mathrm{Tr}[E^B(\Delta')\rho(t + \Delta t | A(t) \in \Delta)]\,\mathrm{Tr}[E^A(\Delta)\rho(t)].\end{aligned} \tag{15}$$

For any Borel set Δ, let $T_\Delta[\rho(t)]$ be the trace class operator defined by

$$T_\Delta[\rho(t)] = \mathrm{Tr}[E^A(\Delta)\rho(t)]\rho(t + \Delta t | A(t) \in \Delta). \tag{16}$$

Then, (16) defines the transformation T_Δ that maps $\rho(t)$ to $T_\Delta[\rho(t)]$. It follows from (15) and (16) that T_Δ satisfies

$$\Pr\{A(t) \in \Delta, B(t + \Delta t) \in \Delta'\} = \mathrm{Tr}[E^B(\Delta')T_\Delta[\rho(t)]]. \tag{17}$$

Suppose that the state $\rho(t)$ is a mixture of the states ρ_1 and ρ_2, i.e.,

$$\rho(t) = \alpha\rho_1 + (1 - \alpha)\rho_2 \tag{18}$$

where $0 < \alpha < 1$. This means that at the time t the measured object $\mathbf{S}$ is sampled randomly from an ensemble of similar systems described by the density operator ρ_1 with probability α and from another ensemble described by the density operator ρ_2 with probability $1 - \alpha$. Thus we have naturally

$$\begin{aligned}&\Pr\{A(t) \in \Delta, B(t + \Delta t) \in \Delta' | \rho(t) = \alpha\rho_1 + (1 - \alpha)\rho_2\}\\ &= \alpha \Pr\{A(t) \in \Delta, B(t + \Delta t) \in \Delta' | \rho(t) = \rho_1\}\\ &\qquad + (1 - \alpha) \Pr\{A(t) \in \Delta, B(t + \Delta t) \in \Delta' | \rho(t) = \rho_2\},\end{aligned} \tag{19}$$

where $\Pr\{E|F\}$ stands for the conditional probability of E given F. From (17) and (19), we have

$$\begin{aligned}&\mathrm{Tr}\left[E^B(\Delta')T_\Delta\left[\alpha\rho_1+(1-\alpha)\rho_2\right]\right]\\&\quad=\mathrm{Tr}\left[E^B(\Delta')\left[\alpha T_\Delta(\rho_1)+(1-\alpha)T_\Delta(\rho_2)\right]\right].\end{aligned} \tag{20}$$

Since B and Δ' are arbitrary, we have

$$T_\Delta\left[\alpha\rho_1+(1-\alpha)\rho_2\right]=\alpha T_\Delta(\rho_1)+(1-\alpha)T_\Delta(\rho_2). \tag{21}$$

Thus, T_Δ is an affine transformation from the space of density operators to the space of trace class operators, and hence it can be extended to a unique positive linear transformation of the trace class operators [10].

We have proved that for any apparatus $\mathbf{A}$ measuring A there is uniquely a family $\{T_\Delta|\ \Delta\in\mathcal{B}(\mathrm{R})\}$ of positive linear transformations of the trace class operators such that (16) and (17) hold, where $\mathcal{B}(\mathrm{R})$ stands for the collection of all Borel sets. This family of linear transformations will be referred to as the *operational distribution* of the apparatus $\mathbf{A}$.

By the countable additivity of probability, if $\Delta=\bigcup_n\Delta_n$ for disjoint Borel sets Δ_n, we have

$$\begin{aligned}&\Pr\{A(t)\in\Delta,B(t+\Delta t)\in\Delta'\}\\&\quad=\sum_n\Pr\{A(t)\in\Delta_n,B(t+\Delta t)\in\Delta'\}.\end{aligned} \tag{22}$$

By (17) and (22) we can prove (14a). Equations (14b) and (14c) are obvious from (16). Thus, the operational distribution $\{T_\Delta|\ \Delta\in\mathcal{B}(\mathrm{R})\}$ satisfies the Davies-Lewis postulate.

Mathematical theory of operational distributions was introduced by Davies and Lewis [9] based on relation (14a) as a mathematical axiom and developed in [11]. Theory of measuring processes based on completely positive operational distributions was developed extensively in [4,12–18].

Now we are ready to state the following important relations for operational distributions.

Theorem 2. *Let $\{T_\Delta|\ \Delta\in\mathcal{B}(\mathrm{R})\}$ be a family of positive linear transformations on $\tau c(\mathcal{H})$ satisfying (14a) and (14b). Then, for any Borel set Δ and any trace class operator ρ we have*

$$\begin{aligned}T_\Delta(\rho)&=T_\mathrm{R}\left(E^A(\Delta)\rho\right)=T_\mathrm{R}\left(\rho E^A(\Delta)\right)\\&=T_\mathrm{R}\left(E^A(\Delta)\rho E^A(\Delta)\right).\end{aligned} \tag{23}$$

A proof of the above theorem was given in [4] for the case where T_Δ is completely positive, and another proof was given in [19] for the case where A is discrete. The general proof necessary for the above theorem is obtained by modifying the above proofs.

By (7), (13), and (16) we have

$$T_\mathrm{R}(\rho)=\mathrm{Tr}[U(\rho\otimes\sigma)U^\dagger] \tag{24}$$

for any $\rho\in\tau c(\mathcal{H})$. From (14c), (23), and (24), we obtain the following characterization of the possible selective state changes: *if* $\Pr\{A(t)\in\Delta\}>0$*, the state* $\rho(t+\Delta t|A(t)\in\Delta)$ *is uniquely determined as*

$$\begin{aligned}
&\rho(t+\Delta t|A(t)\in\Delta)\\
&\qquad=\frac{\mathrm{Tr}_{\mathbf{A}}[U(\rho(t)E^A(\Delta)\otimes\sigma)U^\dagger]}{\mathrm{Tr}[E^A(\Delta)\rho(t)]} &(25a)\\
&\qquad=\frac{\mathrm{Tr}_{\mathbf{A}}[U(E^A(\Delta)\rho(t)\otimes\sigma)U^\dagger]}{\mathrm{Tr}[E^A(\Delta)\rho(t)]} &(25b)\\
&\qquad=\frac{\mathrm{Tr}_{\mathbf{A}}[U(E^A(\Delta)\rho(t)E^A(\Delta)\otimes\sigma)U^\dagger]}{\mathrm{Tr}[E^A(\Delta)\rho(t)]}. &(25c)
\end{aligned}$$

Now, we are ready to prove our main theorem.

Proof of Theorem 1. It suffices to show the equivalence between (11) and (12). First, note that from (15) and (25b), the joint probability distribution of A and B is given by

$$\begin{aligned}
&\Pr\{A(t)\in\Delta, B(t+\Delta t)\in\Delta'\}\\
&\qquad=\mathrm{Tr}\left[E^B(\Delta')\mathrm{Tr}_{\mathbf{A}}[U(E^A(\Delta)\rho(t)\otimes\sigma)U^\dagger]\right]\\
&\qquad=\mathrm{Tr}\left[E^A(\Delta)\mathrm{Tr}_{\mathbf{A}}[U^\dagger(E^B(\Delta')\otimes I)U(I\otimes\sigma)]\rho(t)\right].
\end{aligned} \tag{26}$$

If (11) holds, (12) follows from (26). Conversely, suppose that (12) holds. By substituting $\Delta = \mathrm{R}$ in (12), we have

$$\Pr\{A(t)\in\mathrm{R}, B(t+\Delta t)\in\Delta'\} = \mathrm{Tr}[E^B(\Delta')\rho(t)]. \tag{27}$$

On the other hand, from (26) we have

$$\begin{aligned}
&\Pr\{A(t)\in\mathrm{R}, B(t+\Delta t)\in\Delta'\}\\
&\qquad=\mathrm{Tr}\left[\mathrm{Tr}_{\mathbf{A}}[U^\dagger(E^B(\Delta)\otimes I)U(I\otimes\sigma)]\rho(t)\right].
\end{aligned} \tag{28}$$

Since $\rho(t)$ is arbitrary, from (27) and (28) we obtain (11). Therefore, (11) and (12) are equivalent. To prove A and B commute, suppose that (12) holds. By the positivity of probability, the right hand side is positive. Since $\rho(t)$ is arbitrary, the product $E^A(\Delta)E^B(\Delta')$ is a positive self-adjoint operator so that $E^A(\Delta)$ and $E^B(\Delta')$ commute. Since Δ and Δ' are arbitrary, A and B commute. □

Obviously from (11), if U and $B\otimes I$ commute, i.e.,

$$[U, E^B(\Delta)\otimes I] = 0 \tag{29}$$

for any Borel set Δ, then the A-measurement does not disturb the observable B. However, (29) is not a necessary condition for nondisturbing measurement. In the case where σ is a pure state $\sigma = |\xi\rangle\langle\xi|$, it follows from (11) that the A-measurement does not disturb the observable B if and only if

$$[U, E^B(\Delta)\otimes I]|\psi\rangle|\xi\rangle = 0 \tag{30}$$

for any Borel set Δ and any state vector $|\psi\rangle$ of $\mathbf{S}$.

Consider the case where the system $\mathbf{S}$ consists of two subsystems $\mathbf{S}_1$ with state space $\mathcal{H}_1$ and $\mathbf{S}_2$ with state space $\mathcal{H}_2$ so that $\mathcal{H} = \mathcal{H}_1\otimes\mathcal{H}_2$. Suppose that A belongs to $\mathbf{S}_1$ and B

belongs to $\mathbf{S}_2$ so that $A = C \otimes I$ and $B = I \otimes D$ for some C on $\mathcal{H}_1$ and for some D on $\mathcal{H}_2$. In this case, we say that the A-measurement using the apparatus $\mathbf{A}$ is *local* iff the measuring interaction couples only $\mathbf{S}_1$ and $\mathbf{A}$. Thus, the A-measurement is instantaneous and local if and only if we have

$$[U, I_1 \otimes X \otimes I] = 0, \tag{31}$$

for any operator X on $\mathcal{H}_2$, where I_1 is the identity on $\mathcal{H}_1$ and I is the identity on $\mathcal{H}_{\mathbf{A}}$. From (29) and (31), any local instantaneous measurement of A does not disturb B and any local instantaneous measurement of B does not disturb A either. Therefore, it is concluded that *any pair of local instantaneous measurements of* $A = I \otimes C$ *and* $B = I \otimes D$ *satisfies the joint probability formula*

$$\Pr\{A(t) \in \Delta, B(t) \in \Delta'\} = \mathrm{Tr}[(E^C(\Delta) \otimes E^D(\Delta'))\rho(t)], \tag{32}$$

regardless of the order of the measurement.

In the EPR paper [20], the so called EPR correlation is derived theoretically under the assumption that the pair of measurements satisfies the projection postulate, but the present result concludes that the EPR correlation holds for any pair of local instantaneous measurements as experiments have already suggested.

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